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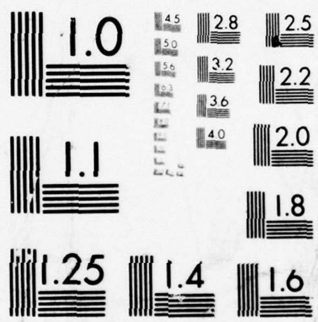

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A CUT APPROACH TO A CLASS OF QUADRATIC INTEGER PROGRAMMING PROBLEMS

Research Report No. 78-2

by

Jean-Claude Picard*
H. Donald Ratliff**

January, 1978

**Department of Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611

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*Federal University of Paraiba, Campina Grande, Brasil

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This paper presents an efficient algorithm for solving a class of quadratic integer programming problems. These problems include discrete versions of the quadratic placement problem and the squared Euclidean distance problem. The algorithm solves a finite sequence of minimum cut problems, or equivalently maximum flow problems, on a graph with n + 2 vertices where n is the number of variables in the problem.

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Abstract

This paper presents an efficient algorithm for solving a class of quadratic integer programming problems. These problems include discrete versions of the quadratic placement problem and the squared Euclidean distance problem. The algorithm solves a finite sequence of minimum cut problems, or equivalently maximum flow problems, on a graph with n + 2 vertices where n is the number of variables in the problem.

Consider an integer program of the form

$$\text{minimize} \quad f(Y) = \sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}\, y_j y_k + \sum_{j=1}^{n} b_j y_j \tag{P0}$$

$$\text{s.t.} \qquad y_j = \ell_j, \ell_j+1, \ldots, u_j \quad j = 1, 2, \ldots, n$$

where ℓ_j and u_j are non-negative integers, $q_{jk} = q_{kj}$, $q_{jk} \leq 0$ for $j \neq k$ and $\sum_{j=1}^{n} q_{jk} \geq 0$ for $j = 1, 2, \ldots, n$. We will develop an algorithm which solves (P0) by solving a sequence of not more than $\sum_{j=1}^{n} (u_j - \ell_j)$ minimum cut problems on a graph with n+2 vertices.

Problems which are of the form (P0) include discrete versions of the squared Euclidean distance location problem (White [6]) and the n-dimensional quadratic placement problem (Hall [1]). The discrete squared Euclidean distance location problem has the form

$$\min \quad \sum_{j=1}^{n} \sum_{k=1}^{n} v_{jk}[(z_j - z_k)^2 + (z_j' - z_k')^2] + \sum_{j=1}^{n} \sum_{k=1}^{m} w_{ji}[z_j - a_i)^2 + (z_j' - a_i')^2]$$

$$\text{s.t.} \quad z_j = \ell_j, \ell_j+1, \ldots, u_j \text{ and } z_j' = \ell_j', \ell_j'+1, \ldots, u_j'$$

where (z_j, z_j') is the location of new facility j, (a_i, a_i') is the location of old facility i, and v_{jk} and w_{ji} are non-negative weights normally corresponding to the number of trips between facilities. This problem decomposes into a problem in z and a problem in z' both of which are easily put into the form (P0). The n-dimensional quadratic placement problem can be considered as a variant of the squared Euclidean distance problem.

The underlying ideas for the algorithm to be presented here are as follows. We note from Picard and Ratliff [2] that a restricted version of (PO), where each y_j can take on only its lower bound ℓ_j or its lower bound plus one, can be efficiently solved as a minimum cut problem on a graph. (This will be illustrated later.) If Y* is the optimum solution to this restricted problem, we will show that there exists an optimum solution Y° to the original problem with $y_j^\circ \geq y_j^*$ for j=1, 2, ..., n. Hence for each j with $y_j^* = \ell_j+1$ we can increase the lower bound on y_j by one. We can then repeat this process until we obtain an optimum solution Y* with each y_j^* equal to its lower bound. We will show that such a solution is optimum to the original problem. Before formalizing the algorithm, we will develop the necessary theoretical underpinning.

Theoretical Results

Consider the problem

$$\begin{aligned} &\text{minimize } f(X) \\ &\quad \text{s.t.} \quad x_j = 0, 1, 2, \ldots, u_j' \quad \text{for } j = 1, 2, \ldots, n \end{aligned} \tag{P1}$$

where $f(\cdot)$ is as defined for (PO) and u_j' is a nonnegative integer for j = 1, 2, ..., n. Also consider the related problem

$$\begin{aligned} &\text{minimize } f(X) \\ &\quad \text{s.t.} \quad x_j = 0 \text{ or } 1 \quad \text{for } j = 1, 2, \ldots, n \end{aligned} \tag{P2}$$

where $f(\cdot)$ is as defined for (PO).

Lemma 1: If X* is an optimum solution to (P2), then there is an optimum solution X' to (P1) with $x_j' \geq x_j^*$.

Proof of Lemma 1: Let X° be any optimum solution to (P1) and let $R = \{j \mid x_j^* = 1$ and $x_j^\circ = 0\}$. Define the vectors X' and X" as

$$x_j' = \begin{cases} 1 \text{ if } j \in R \\ x_j^\circ \text{ otherwise.} \end{cases}$$

$$x_j'' = \begin{cases} 0 \text{ if } j \in R \\ x_j^* \text{ otherwise.} \end{cases}$$

Noting that for each $j \in R$ the variables $x_j^* = 1$ and $x_j' = 0$ we have

$$f(X^*) - f(X'') = \sum_{j \in R} \sum_{k \in R} q_{jk} + 2 \sum_{j \in R} \sum_{k \notin R} q_{jk}\, x_k^* + \sum_{j \in R} b_j \leq 0 \tag{R1}$$

since X" is feasible to (P2) and X* is optimum to (P2). Also noting that for each $j \in R$ the variables $x_j^\circ = 0$ and $x_j' = 1$ we have

$$f(X') - f(X^\circ) = \sum_{j \in R} \sum_{k \in R} q_{jk} + 2 \sum_{j \in R} \sum_{k \notin R} q_{jk}\, x_k^\circ + \sum_{j \in R} b_j \tag{R2}$$

Now since $q_{jk} \leq 0$ and for $k \notin R$ we have $x_j^\circ \geq x_j^*$ it follows that each term of (R2) is less than or equal to the corresponding term of (R1). Hence, $f(X') \leq f(X^\circ)$. Since X' is feasible to (P1) and has $x_j' \geq x_j^*$ the result follows.

Q.E.D.

<u>Lemma 2</u>: If X* is an optimum solution to (P2) and $x_j^* = 0$ for j=1, 2, ..., n, then X* is optimum to (P1).

Proof of Lemma 2: Assume that there exists an X° feasible to (P1) with $f(X^\circ) < 0$. (Note that $f(X^*) = 0$.)

Define the vectors X' and X" as

$$x_j' = \begin{cases} 1 \text{ if } x_j^\circ \geq 1 \\ 0 \text{ if } x_j^\circ = 0 \end{cases}$$

$$x_j'' = x_j^\circ - x_j' \text{ for } j = 1, 2, \ldots, n.$$

Then

$$f(X^\circ) = f(X' + X'') = \sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}(x_j'+x_j'')(x_k'+x_k'') + \sum_{j=1}^{n} b_j(x_j'+x_j'')$$

$$= f(X') + f(X'') + 2 \sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}\, x_j'\, x_k'' < 0 \tag{R3}$$

Since X' is feasible to (P2) and X* is optimum to (P2) with $f(X^*) = 0$, it must be the case that $f(X') \geq 0$.

Now consider the term $2\sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}\, x'_j\, x''_j$. From the assumption that $\sum_{k=1}^{n} q_{jk} \geq 0$ we have $\sum_{k=1}^{n} q_{jk}\, x'_j \geq 0$ if $x'_j = 1$. In addition, whenever $x'_j = 0$ we have $x''_j = 0$. Hence, $2\sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}\, x'_j\, x''_k \geq 0$. Therefore, from (R3) it follows that $f(X'') < 0$.

If we repeat this process, each time setting $X^\circ = X''$, after a finite number of repetitions we must obtain an X'' with $f(X'') < 0$ and $x''_j = 0$ or 1. This contradicts the assumption that X* is optimum to (P2).

Q.E.D.

We need to make one further observation before stating an algorithm. If a problem is in the form (P0), we can make the change of variables $x_j = y_j - \ell_j$ to obtain

$$\min \quad f(X) + L = \sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}\, x_j x_k + \sum_{j=1}^{n} x_j (b_j + 2\sum_{k=1}^{n} \ell_k q_{jk})$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk}\, \ell_j \ell_k \qquad \text{(P3)}$$

$$\text{s.t.} \qquad x_j = 0, 1, \ldots, u_j - \ell_j.$$

Since the constant term does not affect the optimization, (P3) is of the same form as (P1). Note that the only coefficients of (P3) which differ from (P0) are those for the linear terms.

Algorithm

(1) Given a problem of the form (P0), transform it via the change of variables $x_j = y_j - \ell_j$ to the form (P1).

(2) Solve the corresponding problem (P2) as a minimum cut problem on a graph (e.g., Picard and Ratliff [4]) to obtain the solution X*. Let $S = \{j \mid x''_j = 1\}$.

(3) If $X = \phi$ stop. It follows from Lemma 2 and the transformation used to obtain (P3) that $x_j = \ell_j$ for $j = 1, 2, \ldots, n$ is optimum to (P0).

(4) If $S \neq \phi$ set $\ell_j = \ell_j + 1$ for all $j \in S$. It follows from Lemma 1 and the transformation used to obtain (P3) that the new ℓ_j is a valid lower bound for y_j in (P0). For each j such that $\ell_j = u_j$ permanently set $y_j = \ell_j$. Go to step (1).

Since at least one lower bound is increased by one at each iteration, the algorithm terminates after at most $\sum_{j=1}^{n} (u_j - \ell_j)$ iterations.

Example Problem

In order to demonstrate the steps of the algorithm, consider the following example problem:

$$\text{minimize } [y_1, y_2, y_3] \begin{bmatrix} 6 & -4 & -2 \\ -4 & 8 & -4 \\ -2 & -4 & 20 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + [y_1, y_2, y_3] \begin{bmatrix} -6 \\ -8 \\ -4 \end{bmatrix} \qquad \text{(P4)}$$

$$\text{s.t.} \quad y_1 = 0, 1, 2, 3 \quad y_2 = 0, 1, 2 \quad y_3 = 0, 1, 2, 3, 4$$

Note that this problem is already in the form (P1), so no initial change of variables is required.

In order to solve the corresponding problem (P2) we can define an undirected graph with vertices, 0, 1, 2, ..., n+1 and capacities defined as follows (Picard and Ratliff [5]).

$$c_{ij} = -q_{ij}$$

$$c_{j,n+1} = \max\{\sum_{k=1}^{n} q_{jk} + b_j, 0\}$$

$$c_{oj} = \max\{-\sum_{k=1}^{n} q_{jk} - b_j, 0\}$$

If $(S, \bar{S})$ is a minimum capacity cut in the graph separating node 0 and node n+1 with $0 \in S$ and $n+1 \in \bar{S}$, an optimum solution to (P2) is $x_j = 1$ for $j \in S$ and $x_j = 0$ for $j \in \bar{S}$.

For this example the graph is as shown in Figure 1 where the dashed lined indicates a minimum cut. The optimum solution to (P2) is $x_1 = 1$, $x_2 = 1$, and $x_3 = 0$. From step 4 of the algorithm we set $\ell_1 = 1$, $\ell_2 = 1$, $\ell_3 = 0$.

Performing the change of variables (P3) on (P0) with the new lower bounds yields the new linear term coefficients $[-2, 0, -16]^T$. The constant term is ignored and the remaining coefficients are the same as in (P4). The new graph and minimum cut are shown in Figure 2. An optimum solution to the new problem (P2) is $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$. From step 4 we get $\ell_1 = 2$, $\ell_2 = 2$, and $\ell_3 = 1$.

Again performing the change of variables (P3) on (P0) with the new lower bounds yields the new linear term coefficients $[-2, 0, 12]^T$. The new graph and minimum cut are shown in Figure 3. This time the optimum solution to (P2) is $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Hence, an optimum solution to (P0) is $y_1 = \ell_1 = 2$, $y_2 = \ell_2 = 2$, and $y_3 = \ell_3 = 1$.

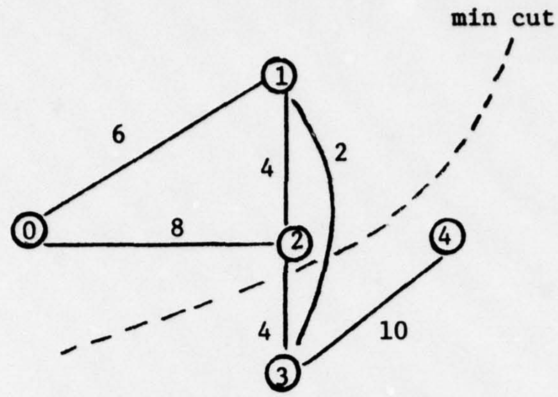

Figure 1: Graph for example iteration one.

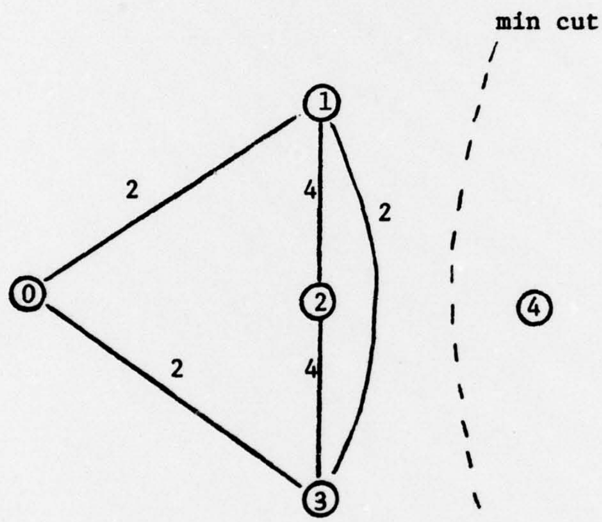

Figure 2: Graph for example iteration two.

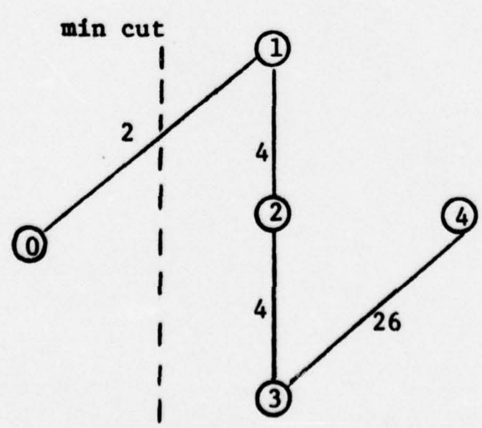

Figure 3: Graph for example iteration three.

Constrained Problems

Constraints of the form $y_j - y_k = d_{jk}$ for d_{jk} integer can be incorporated into (PO) very simply by using a device developed in Hammer, Rosenberg, and Rudeanu [2], and discussed in Hammer and Rudeanu [3, p. 123]. For each constraint of this form we add to f(y) in (PO) the term $(S^+ - S^- + 1)(y_j - y_k - d_{jk})^2$ where S^+ and S^- are the sums of the positive and negative coefficients of f(y), respectively. If the optimum solution to the new problem is feasible to the constraints, it is optimum to the original problem. If the optimum solution to the new problem is not feasible to the constraints, then there does not exist a feasible solution to the original problem. Note that $(S^+ - S^- + 1)$ is positive, hence adding terms of the form $(S^+ - S^- + 1)(y_j - y_k - d_{jk})^2$ to f(y) yields a function with the same form as the original f(y).

By including constraints of this form in the squared Euclidean location problem, we are able to specify the distance between pairs of facilities. This device can be used to model the location of line segments (rectangles for the two dimensional problem) by considering each end of the segment as a point to be located and then constraining the distance between these points to be the length of the segment. This is of particular interest in modeling layout problems.

Conclusions and Extensions

The algorithm presented here can be considered as a generalization of the algorithm in Picard and Ratliff [5] for the rectilinear distance location problem. The basic difference is that additional special structure in the rectilinear problem (i.e., each cut problem solved partitions the variables into two sets and each set can be solved as an independent problem) allows a more efficient algorithm which requires at most, m-1, where m is the number of old facilities, minimum cut problems on a graph with n+2 vertices.

There are several possible generalizations for the procedure. First, the form (P0) applies to a somewhat more general problem than the squared Euclidean distance location problem and hence, may have other applications in a location context. Second, Lemma 1 requires only that $q_{ij} \leq 0$ for $i \neq j$. In addition, the lemma extends directly to an analogous result related to the upper bounds. Therefore, even if the assumption $\sum_{k=1}^{n} q_{ik} \geq 0$ is not satisfied, the result may be used to increase the variable lower bounds and decrease the variable upper bounds. Finally, problem (P1) where $q_{jj} = 0$ for $j = 1, 2, \ldots, n$ and $q_{ij} \leq 0$ for $i \neq j$ can be solved as a single cut problem on an expanded graph. One can simply replace each variable x_j by $\sum_{k=1}^{u'_j} x_{jk}$ with $x_{jk} = 0$ or 1. The resulting problem is of the form (P2) and can therefore be solved as a single minimum cut problem on a graph with $\sum_{j=1}^{n} u'_j + 2$ nodes.

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